Deformed algebras and effective Hamiltonians

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Abstract

Deformed algebras are used to construct effective Hamiltonians for several nonlinear quantum problems. In all the cases here presented, the deformation parameter can be fitted in such a way that the dynamical properties of the physical models are faithfully reproduced, despite the strong algebraic solvability of the corresponding effective Hamiltonians. In particular, recent results concerning effective fermion-boson and boson-boson interaction Hamiltonians defined through $su_q(2)$ and the Higgs algebra -respectively- are reviewed. A new correspondence between a q-oscillator and finite range potentials is also described.

1 Introduction

The aim of this contribution is to review some recent works [1]-[4] showing that deformed Lie algebras (and, in particular, q-deformed ones [5]-[6]) can be used to define effective Hamiltonians for a wide class of quantum nonlinear interactions with outstanding physical interest. In general, it turns out that an appropriate fitting of the deformation parameter allows the introduction of an effective interaction Hamiltonian which is algebraically simpler and, as a consequence, much closer to exact solvability. The excellent behaviour of these deformed models in order to mimic the physical properties of the original Hamiltonians suggests that deformed algebras can be thought of as "elementary" algebraic structures underlying some quantum nonlinear phenomena. We stress that the approach here presented is completely different from the usual "direct q-deformation" procedure in which new models are introduced through the substitution of the symmetries of the standard ones by their q-analogues (see, for instance, [7] and references therein).

Section 2 deals with the equivalence between systems of $su_q(2)$ effective fermions and fermion-boson interaction Hamiltonians of the type

$$H = \omega_f (T_0 + \Omega) + \omega_b B^\dagger B + G \left((T_+)^k B^\dagger + (T_-)^k B \right) \tag{1}$$

with $k = 1, 2, \ldots$, where the (two-level) fermions are represented by the collective pseudospin operators $\{T_0, T_\pm\}$ and are coupled to a one-mode boson field represented by the creation (annihilation) operators $B^\dagger$ (B). In nuclear theory, such models based on the coupling between bi-fermions and bosons are particularly suitable to describe condensation phenomena and transitions from fermionic to bosonic phases (see [8] and references therein). It turns out [1] that such condensation phenomena can be described through the effective $su_q(2)$ quasifermion Hamiltonians of the type

$$H_q = \alpha\, (\tilde{T}_0 + \beta) + \chi(q)\, q^{k\frac{\tilde{T}_0}{2}} \{(\tilde{T}_+)^k + (\tilde{T}_-)^k\}\, q^{k\frac{\tilde{T}_0}{2}} \tag{2}$$

where $\{\tilde{T}_0, \tilde{T}_\pm\}$ are the generators of the $su_q(2)$ algebra [9]

$$[\tilde{T}_0, \tilde{T}_\pm] = \pm \tilde{T}_\pm, \qquad [\tilde{T}_+, \tilde{T}_-] = \frac{q^{2\tilde{T}_0} - q^{-2\tilde{T}_0}}{q - q^{-1}} \tag{3}$$

and the function $\chi(q)$ is fitted in a suitable way in terms of physical constraints. Note that the "undeformed" $q \to 1$ limit of (17) is not (1), so the former model is not a deformation of the latter. We also stress that the most relevant mathematical properties of the latter q-Hamiltonian are direct consequences of the exact solvability of the following class of Hamiltonians [10]-[11]

$$\mathcal{H} = q^{\frac{\tilde{T}_0}{2}}\, (\tilde{T}_+ + \tilde{T}_-)\, q^{\frac{\tilde{T}_0}{2}} + \delta\, q^{2\tilde{T}_0}, \tag{4}$$

since for any real value of the parameter δ, the eigenvalues and eigenvectors of (4) can be explicitly written in terms of q-numbers and q-Krawtchouk polynomials, respectively.

In Section 3 a similar approach is followed for the boson-boson Hamiltonian describing Second Harmonics Generation (SHG)

$$H_{SHG} = \omega a_1^\dagger a_1 + 2\omega a_2^\dagger a_2 + G \left((a_1^\dagger)^2 a_2 + a_1^2 a_2^\dagger \right) \tag{5}$$

whose effective Hamiltonian on the subspace labelled by the conserved total number of photons n is given by [4]

$$H_{Higgs} = \omega\, n + G(\tilde{J}_+ + \tilde{J}_-) \tag{6}$$

where $\{\tilde{J}_+, \tilde{J}_-\}$ are quasifermion operators given by the generators of a cubic deformation of $su(2)$ known as the Higgs algebra [12]-[14]

$$[\tilde{J}_+, \tilde{J}_-] = 2\,\tilde{J}_z\,(1 + 4\beta j(j+1)) - 8\beta(\tilde{J}_z)^3 \tag{7}$$

and the quantum number j labels each irreducible representation. We recall that the Higgs algebra was firstly introduced in [12] as the dynamical algebra generated by the Laplace-Runge-Lenz operators for the Coulomb problem on 2D spaces with constant curvature.

Finally, the Lorek-Ruffing-Wess (LRW) q-oscillator [15] ($q = e^h > 1$)

$$a\,a^\dagger - q^{-2M}\,a^\dagger\,a = 1, \qquad M = 0, 1, 2, \ldots \tag{8}$$

is revisited in connection with the spectrum of finite range potentials [2]. By performing the finite series expansion of the Hamiltonian

$$H = \omega\,a^\dagger\,a \tag{9}$$

in terms of canonical position and momentum operators, this q-oscillator is found to generate a local momentum-dependent interaction whose spectrum exhibits features of a Woods-Saxon potential. Therefore, this (truncated) q-deformation can be interpreted as a kind of algebraic anharmonicity leading to finite range interactions.

2 Fermion-boson interactions through $su_q(2)$

Among the Hamiltonians (1), we shall concentrate on the cases $k = 1$ (Da Providencia and Schütte model (DPS) [16]) and $k = 2$ (an extended Lipkin model [17]). The "exact" polynomial algebras underlying both Hamiltonians have been recently studied in [3].

2.1 Fermion-boson models for condensed phases

The DPS model is a solvable one which exhibits a phase transition and consists of $N = 2\,\Omega$ fermions moving in two single-shells each with degeneracy $2\,\Omega$. The DPS Hamiltonian reads [16]

$$H_1 = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G(T_+ B^\dagger + T_- B), \tag{10}$$

where G is the strength of the interaction and $\{T_0, T_\pm\}$ are the generators of the $su(2)$ algebra of collective fermions:

$$[T_0, T_+] = T_+, \quad [T_0, T_-] = -T_-, \quad [T_+, T_-] = 2\,T_0. \tag{11}$$

The Hamiltonian (10) commutes with the symmetry operator

$$P = B^\dagger B - (T_0 + \Omega). \tag{12}$$

Let us consider a basis $|m_\Omega, n\rangle$ labeled by the eigenvalues of the number operators for fermions and bosons. In this basis the eigenvalues of P are

$$P|m_\Omega, n\rangle = (n - m_\Omega - \Omega)|m_\Omega, n\rangle. \tag{13}$$

In particular, we shall consider the subspace spanned by the states $|m_\Omega, L + m_\Omega + \Omega\rangle \equiv |m_\Omega; L, \Omega\rangle$ which have a fixed eigenvalue L of P

$$P|m_\Omega; L, \Omega\rangle = L|m_\Omega; L, \Omega\rangle. \tag{14}$$

For $L \geq 0$ the quantum number m_Ω can take the values $m_\Omega = -\Omega, -\Omega + 1, ..., \Omega$, and the subspace has dimension $2\Omega + 1$. If $L < 0$, m_Ω takes the values $m_\Omega = -L - \Omega, -L - \Omega + 1, ..., \Omega$, and accordingly, the dimension of the invariant subspace is $2\,\Omega + L + 1$. In this model the eigenvalue L gives us the information concerning the type of condensed phase. The system will be in a *normal* phase when the correlated ground state is the eigenstate of the symmetry operator P with the eigenvalue $L = 0$. If such ground state has $L \neq 0$ we shall have either a bosonic phase (if $L > 0$) or a fermionic one ($L < 0$).

The $k = 2$ case corresponds to the extended Lipkin-Meshkov-Glick (LMG) Hamiltonian [18]

$$H_2 = \omega_f(T_0 + \Omega) + \omega_b B^\dagger B + G((T_+)^2 B^\dagger + (T_-)^2 B), \tag{15}$$

which exhibits the same kind of condensation phenomena [17]. Similarly, the eigenvalue L of the symmetry operator

$$P_{(-)} = B^\dagger B - \frac{1}{2}(T_0 + \Omega), \tag{16}$$

will label the invariant subspaces, whose dimensions will depend on the values of L and Ω.

2.2 Effective $su_q(2)$ Hamiltonians

The quantum algebra $su_q(2)$ is a Hopf algebra deformation of $su(2)$ [9] with commutation rules

$$[\tilde{T}_0, \tilde{T}_\pm] = \pm \tilde{T}_\pm, \quad [\tilde{T}_+, \tilde{T}_-] = [2\,\tilde{T}_0]_q, \tag{17}$$

where the q-number $[x]_q$ is defined by

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh zx}{\sinh z}, \tag{18}$$

and we shall consider that $q = \mathrm{e}^z$ is a real deformation parameter. In this case, the irreducible representation of $su_q(2)$ with dimension $(2\,j + 1)$ is [5, 6]:

$$\begin{aligned} \tilde{T}_0|j, m\rangle &= m\,|j, m\rangle, \\ \tilde{T}_+|j, m\rangle &= \sqrt{[j + m + 1]_q [j - m]_q}\,|j, m + 1\rangle, \\ \tilde{T}_-|j, m\rangle &= \sqrt{[j - m + 1]_q [j + m]_q}\,|j, m - 1\rangle. \end{aligned} \tag{19}$$

In [1] we have considered the effective DPS model given by

$$H_q = \omega_b L + (\omega_b + \omega_f)(\tilde{T}_0 + \Omega) + \chi(q) q^{\frac{\tilde{T}_0}{2}} (\tilde{T}_+ + \tilde{T}_-) q^{\frac{\tilde{T}_0}{2}} \tag{20}$$

where $\chi(q)$ is a scalar function to be fitted. The fact that the dimension of the $su_q(2)$ representation has to coincide with the one of the invariant subspace of the DPS model implies that $j = \Omega$ and $m = m_\Omega$ for the effective $L \geq 0$ model, while for $L < 0$, $j = \Omega + \frac{L}{2}$ and $m = m_\Omega + \frac{L}{2}$.

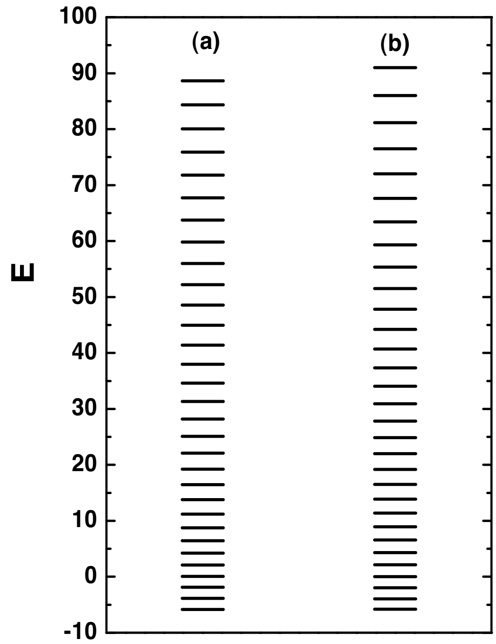

FIGURE 1. Spectra of the Hamiltonians with $2\Omega = 30$, $\omega_f = 1$, $\omega_b = 1$, and $x = G\sqrt{\frac{2\Omega}{\omega_f \omega_b}} = 1.5$. The spectrum denoted by (a) corresponds to the one obtained from the DPS model of Eq. (10), for $L = 5$. The spectrum denoted by (b) is obtained from the effective $su_q(2)$ Hamiltonian of Eq. (20), with $z = 0.02860$. Note that, although the deformation parameter has been fitted in order to reproduce the ground state energy of the DPS model, nearly the full set of eigenstates coincide.

Numerical studies [1] show that (20) reproduces the spectrum and condensation phenomena of (10) with great accuracy provided that q is defined as an appropriate function of both Ω and L (see Figure 1). Thus, the bosonic degrees of freedom included in (10) may be reabsorbed in (20) through the q-deformed fermions.

The extended LMG model (15) can be also approximated by the q-Hamiltonian

$$H_q = \omega_b L + (\omega_f + \frac{1}{2}\omega_b)(\tilde{T}_0 + \Omega) + \chi(q) q^{\tilde{T}_0} ((\tilde{T}_+)^2 + (\tilde{T}_-)^2) q^{\tilde{T}_0}, \tag{21}$$

and, for any set of physical parameters, numerical computations lead to an excellent fitting between the extended LMG model and the effective one [1]. We finally stress that the elimination of the boson field through the definition of an effective quasifermion interaction is a standard procedure in quantum many-body theory. This fact suggests that the simplicity of these q-deformed Hamiltonians can be promising in dealing with more involved fermion-boson interactions like, for instance, the ones arising in QCD models [17] ore those that describe the nonlinear dynamics of an array of two-level Josephson junctions in a resonant cavity [19].

3 SHG and the Higgs algebra

Second harmonics generation is one of the elementary processes in nonlinear quantum optics, and is described by the Hamiltonian

$$H_{SHG} = \omega a_1^\dagger a_1 + 2\omega a_2^\dagger a_2 + G\left((a_1^\dagger)^2 a_2 + a_1^2 a_2^\dagger\right) \tag{22}$$

in which two identical photons with frequency ω interact to form a single photon with frequency $2\,\omega$. This nonlinear phenomenon gives rise to a bunch of nonclassical properties of the radiation field like photon antibunching, squeezing, collapses and revivals, etc. (see [20] and references therein). The implicit equations for the eigenvalues of this Hamiltonian coming from the algebraic Bethe ansatz [21] are not useful for practical calculations, and different approximations are currently used. In this context, effective Hamiltonians for (22) are of interest.

In order to proceed like in the previous section, we firstly realize that the operator

$$P = a_1^\dagger a_1 + 2a_2^\dagger a_2 \tag{23}$$

commutes with H_{SHG}, and thus labels the invariant subspaces with a fixed total number of photons $n = n_1 + 2\,n_2$. Such subspaces are spanned by the states $|n, k\rangle$ with $k = 0, 1, \ldots, \left[\frac{n}{2}\right]$ and their dimension $d(n)$ is given by

$$d \equiv (2j+1) \qquad \text{with} \qquad j = \frac{1}{2}\left[\frac{n}{2}\right]. \tag{24}$$

Within these subspaces, the SHG Hamiltonian is a tridiagonal matrix with constant diagonal entries $\omega\, n$ and upper-diagonal matrix elements given by

$$\langle n, k+1|H_{SHG}|n, k\rangle = \sqrt{(k+1)(n-2k)(n-2k-1)}. \tag{25}$$

3.1 Higgs algebras

Let us consider a $(2j+1)$ dimensional representation space $|j, m\rangle$ and the following deformation of the $su(2)$ algebra representations with deformation parameter β:

$$\begin{aligned}
&\tilde{J}_z|j, m\rangle = m|j, m\rangle \\
&J_\pm^\beta|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)(1 + 2\beta f_\pm(j, m))}|j, m \pm 1\rangle
\end{aligned} \tag{26}$$

where $f_-(j,m) = f_+(j,m-1)$ and the algebra $su(2)$ is recovered under the limit $\beta \to 0$.

If we take

$$f_\pm(j,m) = j(j+1) + m(m \pm 1) \tag{27}$$

we obtain one of the irreducible representations of the algebra

$$\begin{aligned} &[\tilde{J}_z, \tilde{J}_\pm] = \pm \tilde{J}_\pm \\ &[\tilde{J}_+, \tilde{J}_-] = 2\tilde{J}_z + 8\beta \left(\tilde{J}_z\right)^3 \end{aligned} \tag{28}$$

which fully studied in [14]. In general, any cubic deformation of $su(2)$ with no quadratic term in $\tilde{J}_z$ is called a Higgs algebra [12, 14, 13].

In fact, if we consider the slightly different choice

$$f_\pm(j,m) = j(j+1) - m(m \pm 1) \tag{29}$$

we obtain the following Higgs algebra

$$\begin{aligned} &[\tilde{J}_z, \tilde{J}_\pm] = \pm \tilde{J}_\pm \\ &[\tilde{J}_+, \tilde{J}_-] = 2\tilde{J}_z(1 + 4\beta j(j+1)) - 8\beta \left(\tilde{J}_z\right)^3 \end{aligned} \tag{30}$$

in which one of the structure constants depends on the label j of the representation space.

3.2 Higgs quasifermions and SHG

It turns out that for each invariant subspace labeled by $j = j(n)$, the latter algebra can be used to define an effective SHG hamiltonian as follows

$$H_\beta = \omega\, n + G(\tilde{J}_+ + \tilde{J}_-) \tag{31}$$

where $j = \frac{1}{2}\left[\frac{n}{2}\right]$ and the parameter β is fitted in such a way that the ground states of H_{SHG} and H_β coincide. Numerical computations [4] show an excellent agreement between the spectra of both Hamiltonians provided the fitting of the deformation parameter is of the form (see Figures 2 and 3)

$$\beta \approx \frac{\alpha}{j+1} \qquad \text{with} \quad \alpha \approx 2.37. \tag{32}$$

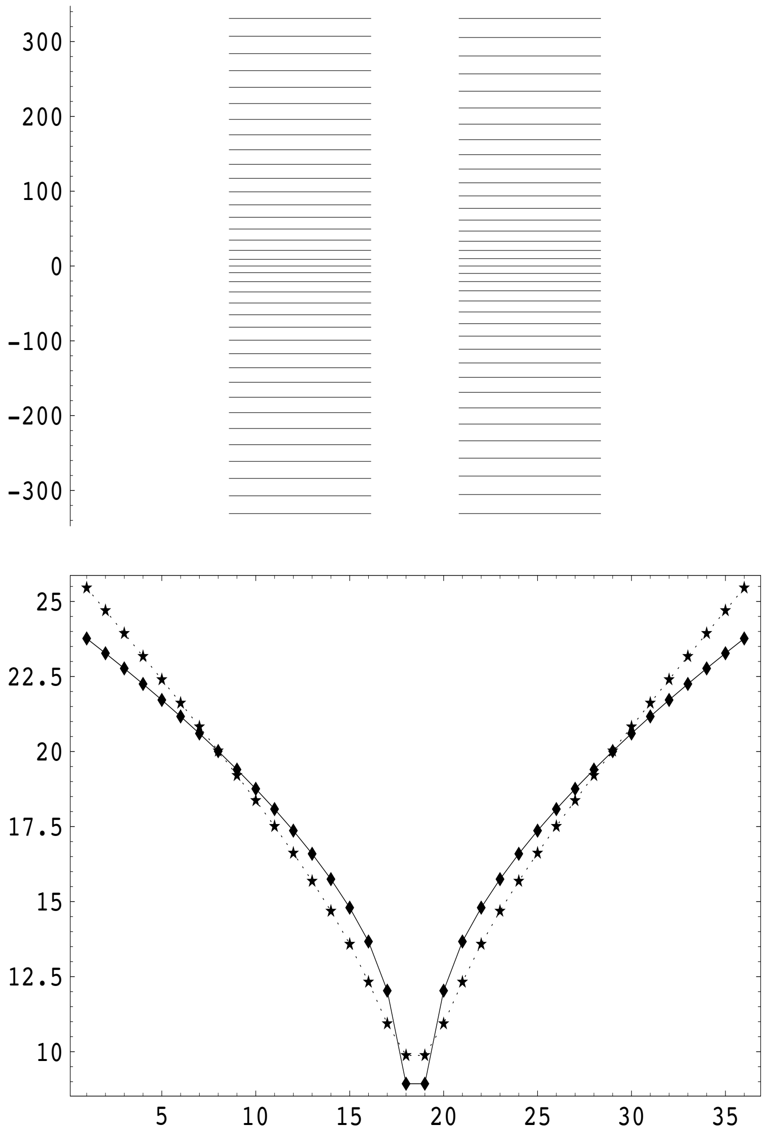

FIGURE 2. Spectra of the SHG Hamiltonian for $j = 18 = n/4$ (left) and of the Higgs model obtained with $\beta = 0.1247$ (right). The figure below plots the differences between consecutive eigenvalues for each model (the continous line corresponds to SHG and the dashed one to the Higgs Hamiltonian). This strong dynamical similarity is preserved for any value of j.

We mention that, by recalling the original work of Higgs [12], this equivalence strongly suggests a connection between effective SHG dynamics and the dynamical algebra of a particle moving on a 2D hyperboloid under the corresponding Coulomb potential. Explicitly, given a subspace j, the same dynamical algebra is obtained for both problems provided that the curvature λ of the 2D hyperboloid is

$$\lambda = -2\,\beta = -2\,\frac{\alpha}{j+1}, \tag{33}$$

and the Coulomb strength μ is defined through the relation

$$\mu^2 = \{(1 + \tfrac{\beta}{2}) + 2\,\beta\, j(j+1)\}(j + \tfrac{1}{2})^2, \tag{34}$$

where we have followed the notation of [13]. Note that both parameters are uniquely defined once the dimension $(2j+1)$ of the SHG subspace is fixed.

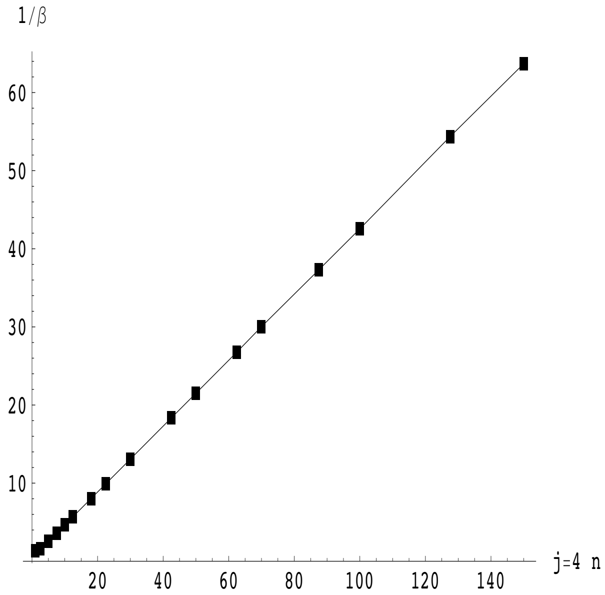

FIGURE 3. Linear fit $\beta^{-1} = \beta^{-1}(j)$ of the Higgs deformation parameter in terms of the $j = 4\,n$ quantum number (n even). All the values of β have been obtained by imposing the effective Higgs model to reproduce the ground state energy of the SHG Hamiltonian.

4 The q-oscillator and finite range potentials

It is well known that the Biedenharn-Macfarlane q-oscillator [22, 23]

$$A\,A^\dagger - q^{1/2}\,A^\dagger\,A = q^{-N/2} \tag{35}$$

is the cornerstone of the boson mapping techniques for quantum algebras and groups [24]-[25]. From the physical point of view, several versions of the q-oscillator have been used to construct new effective anharmonic Hamiltonians related, for instance, with nonclassical states of light [26], vibrational spectra [7] and superfluidity [27].

The Lorek-Ruffing-Wess (LRW) q-oscillator [15]

$$H = \omega\, a^\dagger\, a \tag{36}$$

where

$$a\,a^\dagger - q^{-2M}\,a^\dagger\,a = 1, \qquad M = 0, 1, 2, \ldots \tag{37}$$

is a different version of the q-oscillator that was introduced in the context of the search for a pair of hermitian q-position X and q-momentum P operators fulfilling [15]

$$\begin{aligned} \sqrt{q}X\ P - \frac{1}{\sqrt{q}}P\ X &= iU, \\ U\ X - \frac{1}{q}X\ U &= 0, \\ U\ P - qP\ U &= 0. \end{aligned} \tag{38}$$

These operatos are related to the LRW q-oscillator through

$$\begin{aligned} a &= \alpha\ U^{-2M} + \beta\ U^{-M}P, \\ a^{\dagger} &= \overline{\alpha}\ U^{2M} + \overline{\beta}\ U^{M}P, \end{aligned} \tag{39}$$

where $M = 0, 1, 2, \ldots$ and α and β are complex amplitudes.

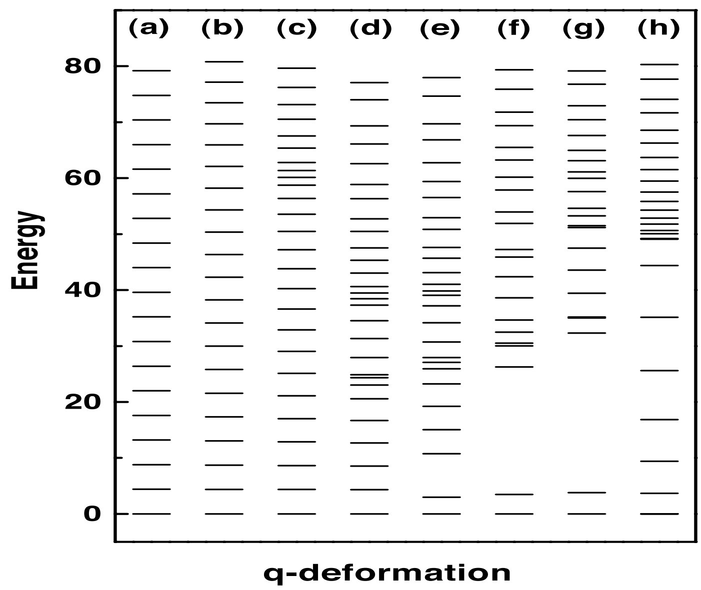

FIGURE 4. Eigenvalues (in arbitrary units) of the harmonic oscillator (case (a)), of the q-deformed harmonic oscillator, cases (b)-(g), and of the Woods-Saxon potential, case (h). The parameters used in the calculations are, respectively: $\hbar\omega = 4.4$ (cases (a)-(g)); and $g = 0.001$ (case (b)), $g = 0.015$ (case (c)), $g = 0.019$ (case (d)), $g = 0.022$ (case (e)), $g = 0.024$ (case (f)), and $g = 0.025$ (case (g)). Case (h) corresponds to the Woods-Saxon potential with $V_0 = -50$, $a = 0.5$, and $R_0 = 1.1$. The calculations have been performed in a basis with sixty five harmonic oscillator shells (see [2] for details).

By considering a realization of (37) in terms the usual position and momentum operators $\{x, p\}$, the leading order of the Hamiltonian (36) can be written as [2]

$$H = H_0 + \frac{1}{4}\sqrt{\frac{hM}{m\omega}}\omega\left[2p - 3m\omega(x^2 p + p\,x^2)\right] \tag{40}$$

where H_0 is the undeformed harmonic oscillator $H_0 = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$. It can be also proven that the leading order Hamiltonian of Equation (15) is, basically, a quartic anharmonic oscillator [2].

The eigenvalues of H, of Eq.(40), for different values of the coupling $g = \frac{\sqrt{hM}}{4}$ are shown in Figure 4. There, for small values of g, cases (b)-(d), the low-energy portion of the spectrum is slightly changed with respect to the pure harmonic oscillator. On the contrary, for larger values of g, cases (e)-(g), the high energy part of the spectrum becomes more dense and it looks similar, case (g), to the spectrum of the finite range Woods-Saxon potential (h)

$$V(r) = \frac{V_0}{1 + e^{\frac{r-R_0}{a_0}}}. \tag{41}$$

Therefore we may conclude that, as a consequence of the q-deformation, the spectrum of the q-deformed harmonic oscillator may approach the spectrum of a finite range potential. We finally remark that the appearance of a gap at low energy, as observed in cases (f) and (g), may be explained by the truncation of the full deformed Hamiltonian (36).

Acknowledgments

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References

[1] A. Ballesteros, O. Civitarese, F.J. Herranz and M. Reboiro, Phys. Rev. **C 66**, 064317 (2002).

[2] A. Ballesteros, O. Civitarese and M. Reboiro, Phys. Rev. **C 68**, 044307 (2003).

[3] A. Ballesteros, O. Civitarese, F.J. Herranz and M. Reboiro, Theor. Math. Phys, **137**, 1495 (2003).

[4] A. Blasco, *Hamiltonian methods in nonlinear Quantum Optics*, M.Sc. Thesis, (Universidad de Burgos, 2002)

[5] V. Chari and A. Pressley, *Quantum Groups* (Cambridge University Press, Cambridge, MA, 1994)

[6] C. Gómez, G. Sierra and M. Ruiz-Altaba, *Quantum Groups in Two-Dimensional Physics* (Cambridge University Press, Cambridge, MA, 1996)

[7] D. Bonatsos and C. Daskaloyannis, Prog. Part. Nucl. Phys. **43**, 537 (1999).

[8] A. Klein and E.R. Marshalek, Rev. Mod. Phys. **63**, 375 (1991).

[9] P.P. Kulish and N. Reshetikhin, Zap. Nauch. Sem. LOMI **101**, 101 (1981).

[10] A. Ballesteros and S.M. Chumakov, J. Phys. A: Math. Gen. **32**, 6261 (1999).

[11] N. M. Atakishiyev and P. Winternitz, J. Phys. A: Math. Gen. **33**, 5303 (2000); N. M. Atakishiyev and A.U. Klymyk, J. Phys. A: Math. Gen. **35**, 5267 (2002).

[12] P.W. Higgs, J. Phys. A: Math. Gen. **12**, 309 (1979).

[13] A.S. Zhedanov, Mod. Phys. Lett. A. **7**, 507 (1992).

[14] B. Abdesselam, J. Beckers, A. Chakrabarti and N. Debergh, J. Phys. A: Math. Gen. **29**, 3075 (1996);
N. Debergh, J. Phys. A: Math. Gen. **30**, 5239 (1997).

[15] A. Lorek, A. Ruffing and J. Wess, Z. Phys. **C 74**, 369 (1997).

[16] D. Schütte and J. Da Providencia, Nucl. Phys. A **282**, 518 (1977).

[17] S. Lerma H., S. Jesgarz, P. O. Hess, O. Civitarese, and M. Reboiro, Phys. Rev. **C 67**, 055209 (2003); *ibid.* , Phys. Rev. **C 67**, 055210 (2003)

[18] H. J. Lipkin, N. Meshkov and A. J. Glick, Nucl. Phys. **62**, 188 (1965).

[19] A. Ballesteros, O. Civitarese, F.J. Herranz and M. Reboiro, Phys. Rev. **B**, in press (2003).

[20] J. Delgado, A. Luis, L.L. Sánchez-Soto and A.B. Klimov, J. Opt. B.: Quantum Semiclass. Opt. **2**, 33 (2000).

[21] B. Jurco, J. Math. Phys. **30**, 1739 (1989).

[22] A.J. Macfarlane, J. Phys. **A 22** 4581 (1989).

[23] L.C. Biedenharn, J. Phys. **A 22** L873 (1989).

[24] L.C. Biedenharn and M.A. Lohe, *Quantum Group symmetry and q-tensor algebras*, World Scientific, Singapore, (1995).

[25] M. Chaichian, J.A. de Azcárraga, P. Presnajder, F. Ródenas, Phys. Lett. **B291**, 411 (1992).

[26] M. Artoni, J. Zang and J.L. Birman, Phys. Rev. **A 47** 2555 (1993).

[27] M.R. Monteiro, L.M.C.S. Rodrigues and S. Wulck, Phys. Rev. Lett. **76** 1098 (1996).